Section 1.6. Exponential, Logarithmic & Inverse Trig Functions

Examples:

1. If $y = 2x - 4$ then $x = \frac{1}{2}y + 2$. In words: the inverse of the function $f(x) = 2x - 4$ is $f^{-1}(x) = \frac{1}{2}x + 2$. (NOTE: $f^{-1}(x) \neq 1/f(x)$.)
2. If $y = x^2 + 4$ then $x = \sqrt{y-4}$ or $x = -\sqrt{y-4}$ so x cannot be recovered from y. In words: there is no inverse of the function $g(x) = x^2 + 4$.

Definition: A function $f(x)$ with domain A is said to be *one-to-one* if, for any $x_1, x_2 \in A$, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

The function of Example 1 is one-to-one; that of Example 2 is not ($g(-1) = g(1)$).

Definition: Suppose $f(x)$ is a one-to-one function with domain A and range B. Then the inverse f^{-1} of f is the function with domain B and range A defined by $f^{-1}(y) = x$ if and only if $f(x) = y$.

Horizontal Line Test: A function $f(x)$ is one-to-one (so f^{-1} exists) if and only if every horizontal line intersects the graph $y = f(x)$ at most once.

(Recall that the Vertical Line Test was a test fro functions.)

Example 1: $y = 2x + 4$

Example 2: $y = x^2 + 4$, $x \geq 0$.

Finding f^{-1}:

1. Write $y = f(x)$
2. Solve for x in terms of y (if possible).
3. Interchange x and y.

Example: If $f(x) = x^2+4$, $x \geq 0$ then 1. $y = x^2+4$; 2. $x = \sqrt{y-4}$; 3. $f^{-1} = \sqrt{x-4}$.

Graphs: For $y = x^2 + 4$

RULE: If $f(x)$ is one-to-one then the graph of $f^{-1}(x)$ is the reflection of the graph of f in the line $y = x$.

Example: Consider $f(x) = (x+2)^3 - 3$. We can actually find f^{-1}. Solve for x: $y = (x+2)^3 - 3$ means $x = (y+3)^{1/3} - 2$ so that $f^{-1}(x) = (x+3)^{1/3} - 2$. This says that f is one to one on the entire real line and has an inverse. Our earlier theorem says that f^{-1} is differentiable everywhere except possibly at $f(-2) = -3$. We can see in this case that the graph of f^{-1} is vertical at (-3,2).

Logarithm Functions. The function

$$f(x) = a^x$$

is decreasing if $0 < a < 1$ and increasing if $1 < a$ and is uninteresting if $a = 1$. The function is 1–1 and we call its inverse $f^{-1}(x) = \log_a x$. Its graph is (case $a > 1$) Properties

1. $\log_a a^x = x \qquad a^{\log_a x} = x$
2. $\log_a xy = \log_a x + \log_a y$
3. $\log_a x/y = \log_a x - \log_a y$
4. $\log_a x^r = r \log_a x$

Special Case If $a = e = 2.7318281828459\ldots$ then we write $\log_e x = \ln x$ and call this logarithm, the natural logarithm.

$$\ln e^x = x \qquad e^{\ln x} = x$$

All Logarithms are related

$$\log_a x = \frac{\ln x}{\ln a}$$

Inverse Trigonometric Functions: The trigonometric functions are decidedly not 1–1. However there are standard restrictions of the trig functions which are 1–1.

$\text{Sin}\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\text{Cos}\theta$	$0 \le \theta \le \pi$
$\text{Tan}\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Cotθ	$0 < \theta < \pi$
Secθ	$0 \leq \theta \leq \pi, \theta \neq \dfrac{\pi}{2}$
Cscθ	$-\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2}, \theta \neq 0$

Each of these functions have an inverse that is useful in trigonometric problems but the ones that are the most useful in calculus are Sinθ and Tanθ which have inverse $\sin^{-1} x = \arcsin x$ and $\tan^{-1} x = arctanx$. Graph the functions and their inverses.